Codes Correcting Limited Patterns of Random Errors Using S-K Metric

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Abstract: *Coding is essential in all communications and in all multi-operation devices, and errors do occur. For error control, the method in vogue is to use code words with redundant digits. The number of redundant digits is determined based on two things − the number of messages and the kind of errors that need to be controlled. For efficient coding the redundant digits have to be kept to the minimum.*

In this paper we introduce the idea of limited error patters while using the code alphabet $Z_q = \{0, 1, 2, ..., q-1\}, \bmod q,$ when $q > 3$. We define limitations of the errors in a position by substitution of the character there by a specified number of other characters, rather than by any other character. This is not possible through Hamming approach, because there a character in an error could be substituted by any other of the q–1 characters. The firm mathematical base is provided by use of a metric from the class of S-K metrics, Hamming metric being one of these.

The paper gives upper bounds on the codeword lengths for various kinds of "random limited error patterns". Examples and discussion bring out the tremendous improvement and generalization of Rao Hamming bound.

Keywords: *Hamming distance, S-K metric, error patterns, error control in codes, bounds.*

1. Introduction

With the advancement of information technology, coding theory is having new challenges. This is because the communication channels, the automata or the electronic devices, where they find use have varying characteristics. The errors patterns differ. Theory of error control coding started simply with binary code words of fixed length, with basically three parameters, namely the number information digits, the check digit and the Hamming distance between n-tuples. The errors considered were the random errors and the burst errors. When studies were extended to q-nary case, these things continued without further necessary refinement. A single error continued to be an error in any position of any magnitude. The cause for this can be traced in the inbuilt nature of Hamming distance. Lee distance [1], yet another distance is also fixed in nature, with limited scope for study. The work progressed mainly in two directions, these related to optimality considerations and construction of codes capable of correcting a certain number of random error correcting codes. Practically all designed distance codes, BCH, Goppa, Helgert, turned out to be "bad", with rate falling with increasing length, that is asymptotically [1].

Since error control coding now is not limited to distant communication only, mathematics was needed which can suitably match the characteristics of the device for which the coding was required and can consider the correction of only those patterns that need to be corrected rather than the wasted capacity of the correcting non-errors by default.

S h a r m a and K a u s h i k [4, 5] investigated the question of other possible different ways of defining distances/metrics (S-K distances/metrics) over $Z_q = \{0, 1, 2, ..., q-1\}, \bmod q$, when $q > 2$. They found a whole class of possible distances defined over Z_q, by considering partitions of Z_q, satisfying certain conditions that we take up in the next section.

This class of S-K distances, beside being rich in choice for properly matching the channel, provides possibilities of handling a wide variety of errors, which was not possible with Hamming distance or even with Lee distance considerations. Hamming and Lee metrics are in fact the particular rather extremal cases of S-K studies, refer to [4].

With nature of the errors vastly different and S-K metric, it is possible to consider codes which will have well defined error characteristics and will be far more efficient than otherwise possible with Hamming distance consideration. Since only those errors patterns of a given weight, that form a part of the total error patterns of that weight, are considered for correction, we call them "codes correcting partial errors". This is undertaken in this paper.

The paper is organized as follows: Section 2 is preliminary, where we recapitulate basic concepts, definitions and results required for later studies. In Section 3 we introduce the idea of partial error correction and obtain lower bounds for a variety of partial error correction. Section 4 gives examples of the partial error correcting codes.

2. Preliminaries

We shall be considering vectors over $Z_q = \{0, 1, 2, ..., q-1\}$, ***the ring of integer, mod q***, where in general in our study, $q > 3$.

    Definition 1. *Hamming Weight and Distance: Given a vector* $u = (a_1, a_2, ..., a_n)$ *over* Z_q***,*** *the* Hamming weight *of vector u is the number of its nonzero components.*

    Next, given two n-vectors $u = (a_1, a_2, ..., a_n)$ *and* $v = (b_1, b_2, ..., b_n)$ *the* Hamming distance *between u and v is the number of components, in which the two n-tuples differ or is the weight of their component-wise difference vector.*

    Definition 2. Sharma-Kaushik (S-K) Partition: Let us consider a partition of Z_q, into (disjoint, nonempty) subsets $B_0, B_1, ..., B_{m-1}$, where m is an integer greater than or equal to 2, such that

(i) $$B_0 = \{0\};$$

(ii) $$\text{for } i \in Z_q, \ i \in B_s \Leftrightarrow q - i \in B_s;$$

(iii) if $i \in B_s$, and $j \in B_t$, and $s > t$ (in the order of their natural occurrence in Z_q)

then
$$\min\{i, q-i\} > \min\{j, q-j\};$$

(iv) if $s > .t$, $|B_s| \geq |B_t|$, except for $s = m-1$, in which case we may have

$$\left|B_{m-1}\right| \geq \frac{1}{2}\left|B_{m-2}\right|,$$

where $|B| =$ number of elements in the set B.

    Note: When $m = 2$, we will call it *Hamming partition* of Z_q. Also when each $|B_s| = 2$, for $0 < s < m-2$, the partition will be called *Lee partition*, since in that case it leads to Lee-metric, refer to B e r l e k e m p [1].

    Definition 3. S-K weights and distances:

    S-K weights: It can be mentioned that in S-K scheme of things, the weights and distances are defined in reference to a S-K partition. Thus for different S-K partitions of Z_q**,** these, in general, will have different values for the same element of Z_q, or for a vector over Z_q.

    We first define the S-K weight of an element $j \in Z_q$ corresponding to a S-K partition $P = \{B_0, B_1, ..., B_{m-1}\}$. Denoting it by $W_P(j)$, it is given by
$$W_P(j) = s \text{ if } j \in B_s, \ 0 \leq s \leq m-1.$$

Next, we define S-K weight $W_P(u)$ of $u=(a_1,a_2...,a_n)$, $a_i \in Z_q$, corresponding to S-K partition $P=\{B_0, B_1, ..., B_{m-1}\}$, as the sum of the class-weights of its components, i.e.,

$$W_P(u) = \sum_{i-1}^n W_p(a_i).$$

S-K Distance between two vectors: Given two n-vectors $u=(a_1,a_2,..,a_n)$ and $v=(b_1,b_2,..,b_n)$, where $a_i, b_i \in Z_q$, the S-K distance between vectors u and v associated with S-K partition P is defined as the sum of the S-K distances between their components, i.e.,

$$d_P(u,v) = \sum_{i=1}^n d_p(a_i, b_i) = \sum_{i=1}^n W_P(a_i - b_i),$$

$$d_P(u,v) = W_P(u-v) = \sum_{i=1}^n d_p(a_i - b_i), \text{ refer to [2]}.$$

3. Partial error patterns

Definition 4. *Partial sets: It may be noted that in defining the S-K partition* $P=\{B_0, B_1,..., B_{m-1}\}$, of Z_q *arranged in a circular order,* B_s *is really the collection of all elements of* Z_q, *on either side of 0, assigned a distance* s *from 0. We can thus call* B_s *a subset or a "partial set"' of* Z_q *at distance* s *from 0.*

More generally, for an arbitrary element $j \in Z_q$ its "partial set at a distance s" is given by $B_s(j) = \{B_s + j\}_q$, the addition being in each element of B_s mod q.

Example 1. *Suppose that we have* $Z_q = \{0, 1, 2, 3, 4, 5, 6\}$, *with* $q = 7$, *and let us consider a S-K partition of* Z_7 *given that*

$$P = \left\{ B_0 = \{0\}, B_1 = \{1,6\}, B_2 = \{2,3,4,5\} \right\}.$$

We see a partial set of weight 1 for 0 is B_1 *and the "partial set of weight 2" for 0 is* B_2.

Further the weight-1 partial set of another element, say of 5, is set of those elements of Z_7, given by $B_1(5) = B_1 + 5 = \{1+5, 6+5\}_7 = \{6, 4\}$, and $B_2(5) = B_2 + 5 = \{2+5, 3+5, 4+5, 5+5\}_7 = \{0, 1, 2, 3\}$.

Note: In this example it may be seen that the other two possible S-K partitions of Z_7 are

$$P_L = \left\{ B_0 = \{0\}, B_1 = \{1, 6\}, B_2 = \{2, 5\}, B_3 = \{3, 4\} \right\}$$

and $P_{\mathrm{H}} = \left\{ B_0 = \{0\}, B_1 = \{1, 2, 3, 4, 5, 6\} \right\}$, respectively giving Lee and Hamming distances on Z_7, in terms of the definitions above. The partial sets of different weights (0, 1, 2 and 3 in the case of P_{L} and of weight 0, 1 in the case of P_{H}), the elements in P_{H} and P_{L}, as can be seen are differently defined.

Definition 5. Patterns of limited errors. It may be easily realized that errors follow a pattern, which differs from a system to system. A system in which an entry in a code word is received as another symbol, the error may be called "substitution error". With S-K scheme of things, it is possible to consider various different limited kinds of substitution errors that were not possible under Hamming scheme of things. Some of the possible ones are illustratively detailed below:

1. *Single the error or e-random errors, in which an entry j is substituted by an element of $B_1 + j$* or some *e*-positions $j_1, j_2, ..., j_t$ are similarly substituted correspondingly by an entry with their sum with $B_1 + j_1, B_1 + j_2, ..., B_1 + j_e$. We may call a code correcting *e* such errors "*e* B_1 -error correcting code", and it must have a minimum distance of $2e + 1$, caused by B_1 added substitution positions.

2. *Single error or e-random errors, in which an entry j is substituted by an element of $B_2 + j$* or some *e*-positions are similarly substituted by an entry with their sum with B_2.

One can easily consider different kinds of substitution errors following modifications depending on the partial set (s) that determines the error patterns.

In obtaining a bound on a necessary number of parity-checks for an e-error correcting code, it is customary to define a volume of a sphere of radius e around every code word and consider their mutual disjointness, etc. In the S-K study that we undertake, this idea can be looked more closely and, given an n-vector u, we can find numbers of patterns which have a specified S-K distance from the vector u.

In the theorems that follow, we consider lower bounds on the number of parity check digits, $r = n - k$.

Theorem 1. Given a S-K partition $P = \left\{ B_0, B_1, ..., B_{m-1} \right\}$ of Z_q, q prime, a necessary condition for the existence of an (n, k) code over Z_q, correcting an error in *e*-positions of each of weight one is given by

$$\sum_{t=0}^{e} \binom{n}{t} |B_1|^t \le q^{n-k} \quad \text{or} \quad n - k \ge \log_q \left[\sum_{t=0}^{e} \binom{n}{t} |B_1|^t \right].$$

P r o o f: It may be recalled that the elements in B_1 are of S-K weight 1 and that given an *entry j, an error of weight 1 is obtained by substituting it by an element of $B_1 + j$*. This can happen in $|B_1|$ ways.

Thus, given a code word, the number of vectors at a distance 1 from it are $\binom{n}{1} |B_1|$.

Similarly, the error patterns at a distance 2 are $\binom{n}{2}|B_1|^2$, and those at a distance e are $\binom{n}{e}|B_1|^e$.

Thus, the total number of vectors that are at a distance of e or less are from a vector u of length n:

$$\binom{n}{e}|B_1|^e + \binom{n}{e-1}|B_1|^{e-1} + \binom{n}{e-2}|B_1|^{e-2} + ... + \binom{n}{1}|B_1| + 1 \text{ or } \sum_{t=0}^{e}\binom{n}{t}|B_1|^t.$$

In order that the code is able to correct e-errors of S-K weight 1 each, these vectors around any code word should be disjoint, that is for (n, k) code of q^k code words,

$$q^k \sum_{t=0}^{e}\binom{n}{t}|B_1|^t \le q^n,$$

$$\text{or } \sum_{t=0}^{e}\binom{n}{t}|B_1|^t q^{n-k}.$$

This proves the result.

Note: It may be worthwhile to reflect on the idea of partial sets of certain weights around elements of Z_q. Let us consider the keyboard of a computer, it has keys for various numbers and other symbols. Imagine punching a number or an alphabet key on it. While word processing, one may erroneously strike a key on one or two positions on either side of the right key, rather than any key on the keyboard. These likely positions will constitute the partial set of errors for the number or the symbol key pressed.

Next we want to consider e-random errors, where the substitution errors are caused by elements at distances of 1 *or* 2, *that is from two segments of partial errors are handled for correction.*

First we obtain the result in

Lemma 1. Given a S-K partition $P = \{B_0, B_1, ..., B_{m-1}\}$ of Z_q, q prime, and a vector of length n, the number of vectors obtained by substituting in e-positions, each position j by elements of $B_1 + j$ or $B_2 + j$, is given by

$$\left(|B_1| + |B_2|\right)^e.$$

P r o o f: Let us consider that out of e positions selected, k of them have entry j that is substituted by an entry from $B_1 + j$ and the other $e-k$ by entries from $B_2 + j$.

Then this can be done in $\binom{e}{k}|B_1|^k |B_2|^{e-k}$ ways.

Thus the total, as k varies from 0 to e, gives the required number of vectors as

$$\left(|B_2|^e + \binom{e}{1}|B_1||B_2|^{e-1} + \binom{e}{2}|B_1|^2|B_2|^{e-2} + ... + \binom{e}{k}|B_1|^k|B_2|^{e-k} + ... + /B_1/^e \right)$$

or $\left(|B_1| + |B_2| \right)^e$.

This proves the result.

Note: It can immediately be seen in place of B_1 and B_2, that the result will hold for any two distinct or non-distinct partial sets B_s and B_t, Also it can be generalized from just two to any numbers of partial-sets. We state the result for any s partial-sets below:

Lemma 2. Given a S-K partition $P = \{B_0, B_1, ..., B_{m-1}\}$ of Z_q, q prime, and a vector of length n, the number of vectors obtained by substituting in e-positions, each position j by elements of k partial sets $B_{r_1} + j$, $B_{r_2} + j$, ..., $B_{r_s} + j$, $s \le m - 1$, is given by

$$\left(|B_{r_1}| + |B_{r_2}| + ... + /B_{r_s}/ \right)^e.$$

Theorem 2. Given a S-K partition $P = \{B_0, B_1, ..., B_{m-1}\}$ of Z_q, q prime, a necessary condition for the existence of an (n, k) code over Z_q, correcting the error in e or less positions so that in these positions, the entry j when an error is received as an element of $B_1 + j$ or $B_2 + j$ is given by

$$\sum_{t=0}^{e} \binom{n}{e} \left(|B_1| + |B_2| \right)^t \le q^{n-k}.$$

P r o o f: Now e positions of out n can be chosen in $\binom{n}{e}$ ways and by the Lemma 2 given, these can undergo defined changes in $\left(|B_1| + |B_2| \right)^e$ ways.

Thus, the number of ways any *exactly e* defined errors can occur, is

$$\binom{n}{e} \left(|B_1| + |B_2| \right)^e.$$

The total number of the error patterns in e or less positions is then given by

$$\sum_{t=0}^{e} \binom{n}{t} \left(|B_1| + |B_2| \right)^t.$$

With this number of error patterns around each code, the word to be disjoint, we get the necessary condition as

$$\sum_{t=0}^{e} \binom{n}{t} \left(|B_1| + |B_2| \right)^t \le q^{n-k}.$$

Note: Hamming bound case: In the Hamming case, the partition of Z_q is two partial sets $B_0 = \{0\}$, and $B_1 = \{0, 1, ..., q-1\}$, so that $/ B_1 /= q-1$, and the result of these theorems reduces to the famous Rao Hamming bound [3].

Example 2. Let $Z_q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $q = 13$; to illustrate the results of the above two theorems we consider two S-K partitions of Z_q.

Case A. Let us first consider the S-K partition given by
$$P_1 = \{B_0, B_1, B_2, B_3\},$$
where
$$B_0 = \{0\}, B_1 = \{1, 12\}, B_2 = \{2, 3, 4, 9, 10, 11\}, \text{ and } B_3 = \{5, 6, 7, 8\}.$$

For an (n, k) code correcting single error of weight 1, from Theorem 1, we have the bound
$$1 + \binom{n}{1}|B_1| \le q^{n-k}, \text{ where } n-k = r$$

$$\text{or } n \le \frac{q^r - 1}{|B_1|}.$$

For different values of r, here $|B_1| = 2$, the upper-bound on n for (n, k), $k = n - r$, is given in Table 1. In the table, we have compared the limited $|B_1|$ single error correction bound by a Hamming single error correcting bound. The gains are quite astonishing, for example for $r = 3$, these results tell us that the numbers of messages for $|B_1|$ error correction can use 13^{1095} messages while under -Hamming criterion, this number is a small fraction 13^{180}, and this proportion increases exponentially.

For calculation of the rate, we have used the usual ratio, $R = \dfrac{k}{n}$.

Table 1

r	Upper bound on n with a S-K single error			Upper bound on n with a Hamming-single error		
	n	k	R	N	k	R
1	6	5	0.83333	1	0	0
2	84	82	0.97619	14	12	0.85714
3	1098	1095	0.99727	183	180	0.983606
4	14280	14276	0.99972	2830	2826	0.998586
5	185646	185641	0.99997	30941	30936	0.9998384
6	2413404	2413398	0.999998	402234	402228	0.9999850
7	31374258	31374251	0.9999998	522943	5229036	0.999998661

Case B. Next we consider, for the S-K partition P_1, bounds on n for code correcting errors in a single position which are either of S-K weight 1 or S-K weight 2. From Theorem 2 this bound is given by

$$1 + \binom{n}{1}\left(|B_1| + |B_2|\right) \le q^{n-k} \text{ or } n \le \frac{q^r - 1}{\left(|B_1|/+/B_2|\right)}.$$

In the example when $q = 13$, for different values of r here $\left(|B_1| + |B_2|\right) = 8$, the upper-bound on n for (n, k), $r = n - k$, is given in Table 2.

Table 2

r	Upper bound on n with S-K single error			Upper bound on n with Hamming-single error		
	n	k	R	n	k	R
1	1	0	0	1	0	0
2	21	19	0.90476	14	12	0.85714
3	274	271	0.989051094	183	180	0.983606
4	3570	3566	0.998879551	2830	2826	0.998586
5	46411	46406	0.999892266	30941	30936	0.9998384
6	150837	150831	0.999960222	402234	402228	0.9999850
7	1960891	1960884	0.99999643	522943	5229036	0.999998661

Here again, the thing to be observed is the number of messages with the same number of parity checks, correcting the limited patterns.

Example 3. Next let us consider the S-K partition of Z_q given by

$$P_2 = \{B_0, B_1, B_2, B_3\},$$

where

$$B_0 = \{0\}, \ B_1 = \{1, 2, 11, 12\} \ \text{and} \ B_2 = \{3, 4, 9, 10\}, \ B_3 = \{5, 6, 7, 8\},$$

now for an (n, k) code correcting single error of weight 1, then (by Theorem 1) we have

$$1 + \binom{n}{1}|B_1| \le q^{n-k}, \text{ where } n - k = r \text{ or } n \le \frac{q^r - 1}{|B_1|},$$

and the numerical values are as given in Table 3.

Table 3

r	Upper bound on n with S-K single error			Upper bound on n with a Hamming-single error		
	n	k	R	n	k	R
1	3	2	0.666666	1	0	0
2	42	40	0.952380952	14	12	0.85714
3	549	546	0.994535519	183	180	0.983606
4	7140	7136	0.999439775	2830	2826	0.998586
5	92823	92818	0.999946134	30941	30936	0.9998384
6	1206702	1206696	0.999995027	402234	402228	0.9999850
7	15687129	15687122	0.999999553	522943	5229036	0.999998661

Suppose that we want to correct an error in 1 position of weight 1 and weight 2, then (by Theorem 2) we have

$$1 + \binom{n}{1}\left(|B_1| + |B_2|\right) \le q^{n-k} \quad \text{or} \quad n \le \frac{q^r - 1}{\left(|B_1| + |B_2|\right)},$$

for different values of r, the upper-bound on n for (n, k), $r = n - k$, is given in Table 4.

Table 4

r	Upper bound on n with S-K single error			Upper bound on n with a Hamming-single error		
	n	k	R	n	k	R
1	1	0	0	1	0	0
2	21	19	0.90476	14	12	0.85714
3	274	271	0.989051094	183	180	0.983606
4	3570	3566	0.99887955	2830	2826	0.998586
5	46411	46406	0.99989227	30941	30936	0.9998384
6	150837	150831	0.999960222	402234	402228	0.9999850
7	1960891	1960884	0.99999643	522943	5229036	0.999998661

Example 4. Here we shall illustrate some numerical bounds on n when instead of a single error, the errors in any two or less positions are considered of weight 1 and then again either of weight 1 or 2.

Let $Z_q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $q = 13$, consider S-K partition of Z_q, given by

$$P = \{B_0, B_1, B_2, B_3\},$$

where

$$B_0 = \{0\}, \quad B_1 = \{1, 2, 11, 12\}, \quad B_2 = \{3, 4, 9, 10\} \text{ and } B_3 = \{5, 6, 7, 8\}.$$

Case A. Suppose that we want to correct the error in two positions of weight 1, then (by Theorem 1) we have

$$1 + \binom{n}{1}|B_1| + \binom{n}{2}|B_1|^2 \le q^{n-k},$$

$$n|B_1| + \frac{n(n-1)}{2}|B_1|^2 \le q^r - 1,$$

$$4n + 8n(n-1) \le q^r - 1,$$

$$4n(2n-1) \le q^r - 1 \quad \text{or} \quad n(2n-1) \le \frac{q^r - 1}{4}.$$

This is a quadratic Diophantine in-equation. By trial it is quite possible to find the largest value of n satisfying this inequality. For different values of r, the upper-bound on n for (n, k), $r = n - k$, is given in Table 5.

Table 5

r	Upper bound on n with S-K single error $\left(n\left(2n-1\right)\leq\dfrac{q^r-1}{4}\right)$			Upper bound on n with a Hamming-single error $\left(n(6n-5)\leq\dfrac{q^r-1}{12}\right)$		
	n	k	R	n	k	R
1	1	0	0	1	0	0
2	4	2	0.5	2	0	0
3	16	13	0.8125	5	2	0.4
4	60	56	0.93333	20	16	0.8
5	215	210	0.976744186	72	67	0.93056
6	777	771	0.992277992	259	253	0.97683
7	2800	2793	0.9975	933	926	0.9925

Case B. Next, let us consider the errors in two positions of weight 1, then (by Theorem 1) we have

$$1+\binom{n}{1}\left(\left|B_1\right|+\left|B_2\right|\right)+\binom{n}{2}\left(\left|B_1\right|+\left|B_2\right|\right)^2\leq q^{n-k},$$

$$1+8n+32n\left(n-1\right)\leq q^{n-k},$$

$$8n+32n\left(n-1\right)\leq q^{n-k}-1 \quad\text{or}\quad 8n\left(4n-3\right)\leq q^{n-k}+1,$$

$$n\left(4n-3\right)\leq\frac{q^{n-k}+1}{8},$$

for different values of r, the upper-bound on n for (n, k), $r = n - k$, is given in Table 6.

Table 6

r	Upper bound on n with S-K single error $\left(n\left(2n-1\right)\leq\dfrac{q^r-1}{4}\right)$			Upper bound on n with a Hamming-single error $\left(n(6n-5)\leq\dfrac{q^r-1}{12}\right)$		
	n	k	R	n	k	R
1	1	0	0	1	0	0
2	2	0	0	2	0	0
3	8	5	0.625	5	2	0.4
4	30	26	0.86666	20	16	0.8
5	108	103	0.953704	72	67	0.93056
6	388	382	0.984536	259	253	0.97683
7	443	436	0.984199	933	926	0.9925

4. Concluding remarks

In this paper we have introduced the idea of limited patterns of errors and have obtained results on the upper bounds on word lengths for specified limited patterns of errors. The use of S-K partitions and distances provides a choice of metrics matching the channel characteristics.

It seems quite possible to extend this study to other bounds and to constructions of codes correcting limited patterns of errors.

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